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Chebyshev Approximation by First Degree Rationals on 2-space

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The uniqueness problem for Chebyshev approximation on compact subsets of 2-space by the family of ratios of constants to first degree polynomials is studied.

Let X be a compact subset of real 2-space. We denote a point x of X by a real 2-tuple (y, z). Let $C(X)$ be the space of continuous functions on X, and for $g \in C(X)$ define

$$\| g \| = \sup\{| g(x)| : x \in X\}.$$

Let

$$R(A, x) = P(A, x)/Q(A, x) = a_0/(a_1 + a_2 y + a_3 z).$$

Let $f \in C(X)$. Then the rational Chebyshev approximation problem is to find a coefficient vector A^* minimizing

$$e(A) = \| f - R(A, \cdot)\|$$

under the constraint $Q(A, x) > 0$ for every $x \in X$. Such a coefficient vector A^* is called best and $R(A^*, \cdot)$ is called a best approximation.

It is easily deduced from the results of Brosowski [1, 178–179] and Mairhuber [4, 230–232] that for any nontrivial rational approximation problem on two-dimensional X, there exists $f \in C(X)$ with a nonunique best approximation. However, the f of Brosowski is nondifferentiable, so the uniqueness problem is still open for differentiable f. We obtain a uniqueness result related to one of Collatz [4, 237] for linear approximation.

THEOREM 1. *Let X be strictly convex and let f have first partial derivatives on X. There is at most one best approximation to f on X.*

Proof. Suppose B and C are best coefficient vectors, $R(B, \cdot) - R(C, \cdot) \not\equiv 0$. As the set $\mathcal{A}^*$ of best coefficient vectors is convex [1, Lemma 4], we can assume that B, C are in the interior of $\mathcal{A}^*$. Define

$$M(B) = \{x\colon |f(x) - R(B, x)| = e(B)\}.$$

By continuity of $|f - R(B, \cdot)|$ and compactness of X, $M(B)$ is nonempty. By [1, Lemma 7],

$$R(B, x) = R(C, x), \qquad x \in M(B). \tag{1}$$

The first possibility is that $b_0 = 0$, in which case (1) implies that $C_0 = 0$ and $R(B, \cdot) - R(C, \cdot) = 0$. The second possibility is that $b_0 \neq 0$. Let x_1, x_2 be two distinct points of X, then there exist coefficients d_1, d_2, d_3 such that

$$Q(D, x_i) = \operatorname{sgn}[f(x_i) - R(B, x_i)]/R(B, x_i), \qquad i = 1, 2$$

or

$$R(B, x_i)\, Q(D, x_i) = \operatorname{sgn}[f(x_i) - R(B, x_i)], \qquad i = 1, 2.$$

It follows by the characterization theorem [2, 159] that if $M(B) \subset \{x_1, x_2\}$, B is not best. Hence, $M(B)$ contains at least three points. As $\operatorname{sgn}(R(A, \cdot)) = \operatorname{sgn}(a_0)$ for $Q(A, \cdot) > 0$, we have by (1), $\operatorname{sgn}(b_0) = \operatorname{sgn}(c_0) \neq 0$. Let $\sigma = \operatorname{sgn}(b_0)$. As $R(\alpha A, \cdot) = R(A, \cdot)$ for $\alpha > 0$, we can assume without loss of generality that $R(B, \cdot)$, $R(C, \cdot)$ are normalized so that $b_0 = c_0 = \sigma$. We have then

$$\frac{\sigma}{b_1 + b_2 y + b_3 z} = \frac{\sigma}{c_1 + c_2 y + c_3 z}, \qquad (y, z) \in M(B). \tag{2}$$

Let x_1, x_2, x_3 be distinct points of $M(B)$, then the above equation becomes

$$(b_1 - c_1) + (b_2 - c_2)\, y_i + (b_3 - c_3)\, z_i = 0, \qquad i = 1, 2, 3.$$

If $b_2 - c_2 = 0$ and $b_3 - c_3 = 0$, then $b_1 - c_1 = 0$ and $B \equiv C$. Hence, at least one of $b_2 - c_2$, $b_3 - c_3$ is nonzero.

$$(b_1 - c_1) + (b_2 - c_2)\, y + (b_3 - c_3)\, z = 0$$

is the equation of a straight line in 2-space, hence, the points x_1, x_2, x_3 fall on a line. As X is strictly convex, one of the three points is in the interior of X. Assume it is x_2. It is a maximum or a minimum of $f - R(B, \cdot)$ and by (1) a maximum or minimum of $f - R(C, \cdot)$. We have

$$\begin{aligned} 0 &= \frac{\partial}{\partial y}\left(f(y, z) - \frac{\sigma}{b_1 + b_2 y + b_3 z}\right)\Big]_{y=y_2, z=z_2} \\ &= f_y(y_2, z_2) - \frac{\sigma b_2}{Q^2(B, x_2)}. \end{aligned}$$

Expanding other partial derivatives and knowing that by (2), $Q(B, x_2) = Q(C, x_2)$, we obtain $b_2 = c_2$, $b_3 = c_3$. It follows that $b_1 = c_1$ and $B = C$.

If X is not strictly convex, we may not have uniqueness.

EXAMPLE. Let $X = [0, 1] \times [0, 1]$ and T_2^* be the second Chebyshev polynomial on [0, 1]. Let $G = 1$, $H(y, z) = 1/(1 + z)$, $Z = \{(0, 0), (1/2, 0)(1, 0)\}$, $s(y, z) = T_2^*(y)$. There is no rational $R(A, \cdot)$ with $Q(A, \cdot) > 0$ such that $\operatorname{sgn}(R(A, x) - G(x)) = s(x)$, $x \in Z$. By the definition of zero-sign compatibility and the proof of [3, Theorem 2], the function

$$\begin{aligned} f(y, z) &= 1 + T_2^*(y)[1/2 - |\, 1 - (1/(1 + z))|] \\ &= 1 + T_2^*(y)[(1/(1 + z)) - (1/2)] \end{aligned}$$

has G and H as best approximations.

Similar arguments can be used to show nonuniqueness for all convex regions X with a side parallel to an axis.

The arguments used to obtain Theorem 1 can be used to obtain

THEOREM 2. *Let X be convex and let f have first partial derivatives on X. If f has a nonunique best approximation, then there is a best approximation $R(B, \cdot)$ such that $M(B)$, the error extrema of $R(B, \cdot)$, is contained in a line segment on the boundary of X.*

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